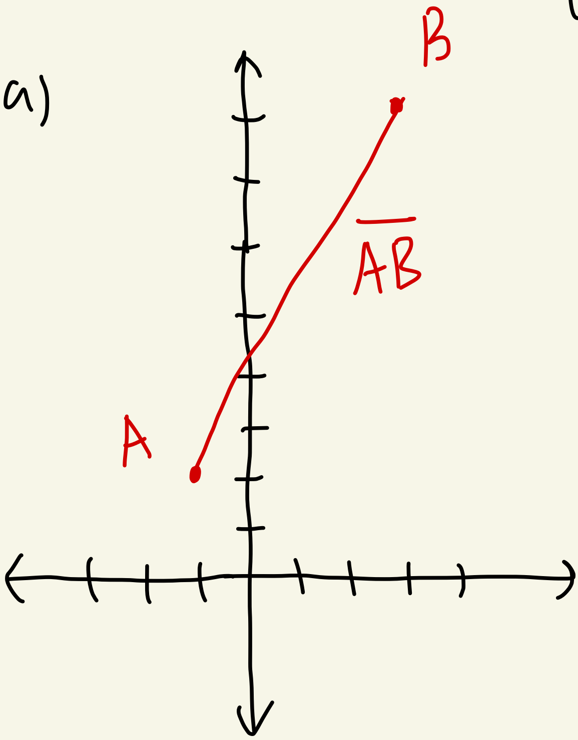
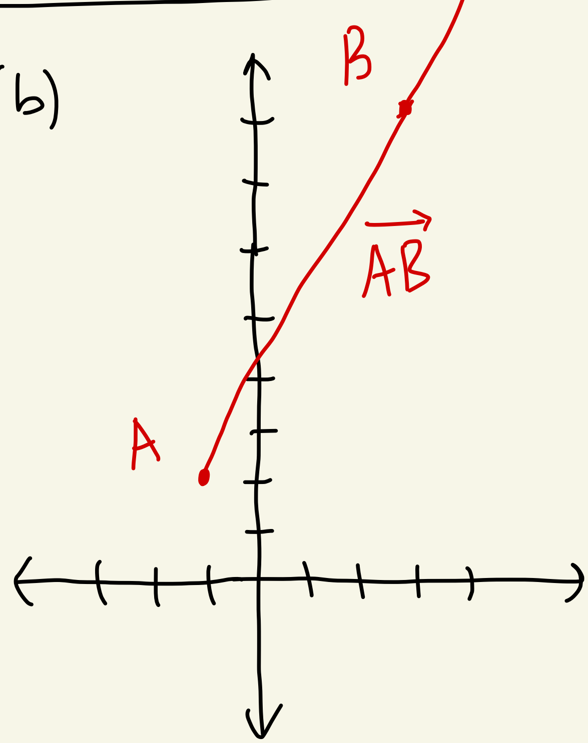
Math 4300
Homework #5
Solutions

① $A = (-1, 2)$, $B = (3, 8)$ in the Euclidean plane

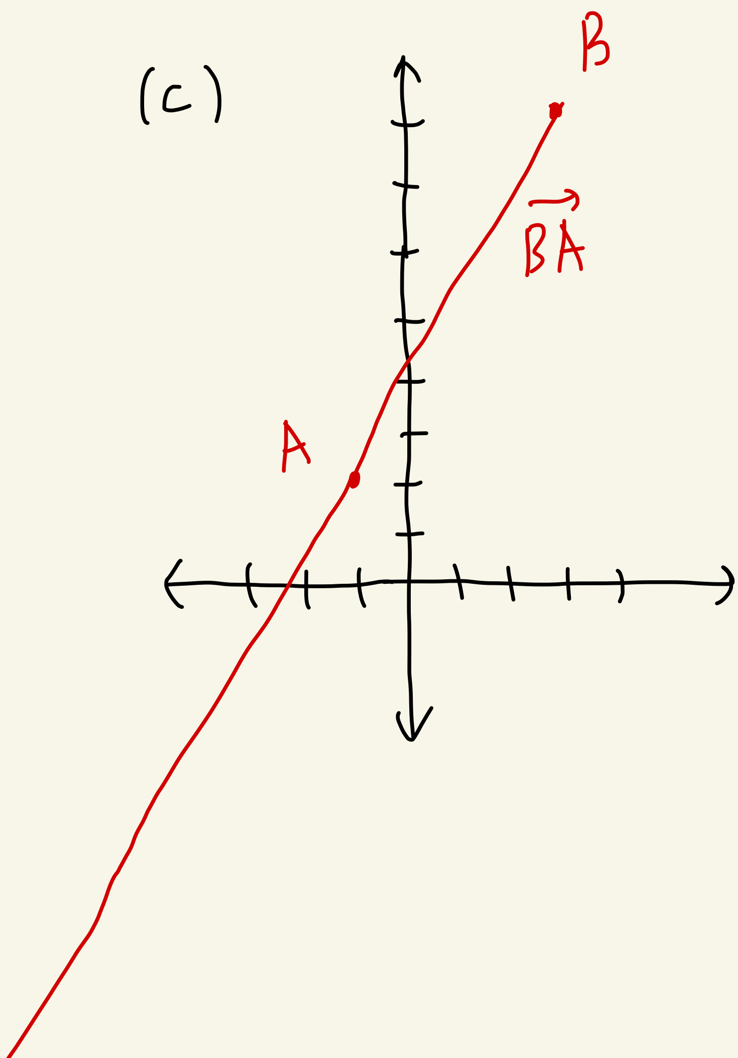
(a)



(b)



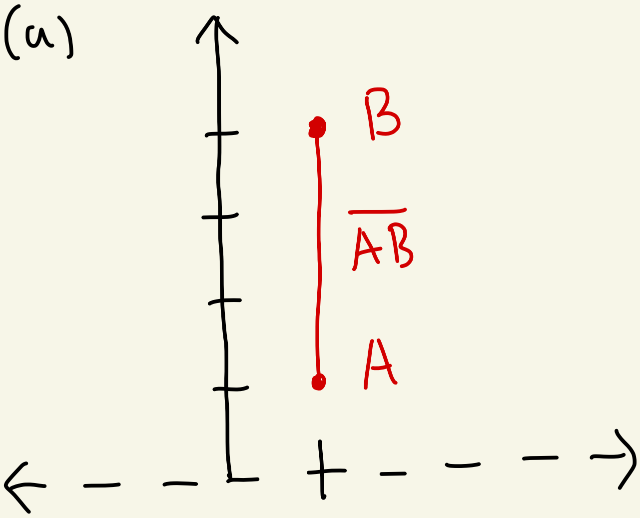
(c)



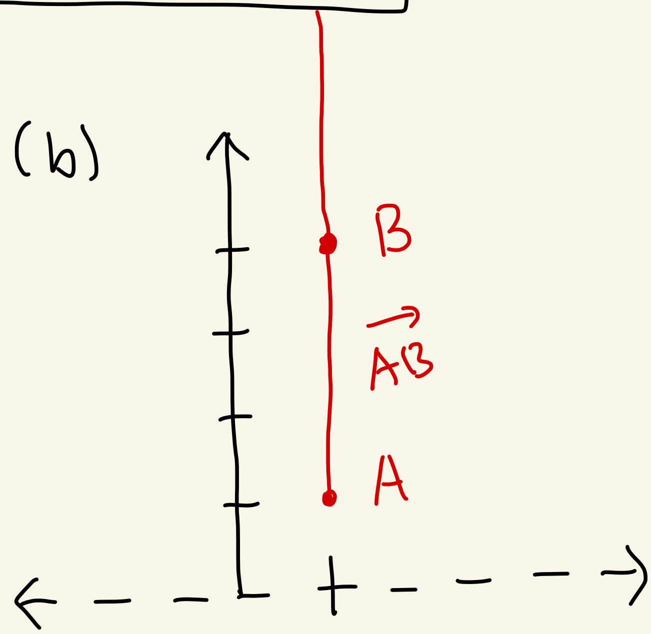
$\boxed{②}$

$$\boxed{A = (1,2),\ B = (1,4) \text{ in the hyperbolic plane}}$$

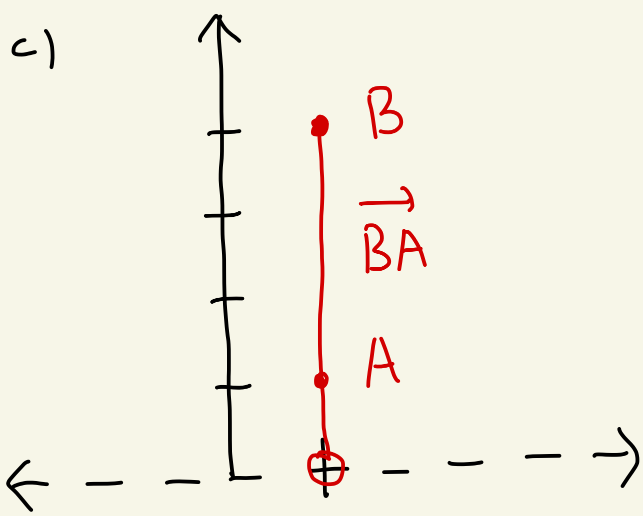
(a)



(b)



(c)



3

$A = (1,2)$, $B = (3,4)$ in the hyperbolic plane

What is the line that A, B lie on?

Plug A and B into $(x-c)^2 + y^2 = r^2$ to get

$$(1-c)^2 + 2^2 = r^2 \quad (1)$$
$$(3-c)^2 + 4^2 = r^2 \quad (2)$$

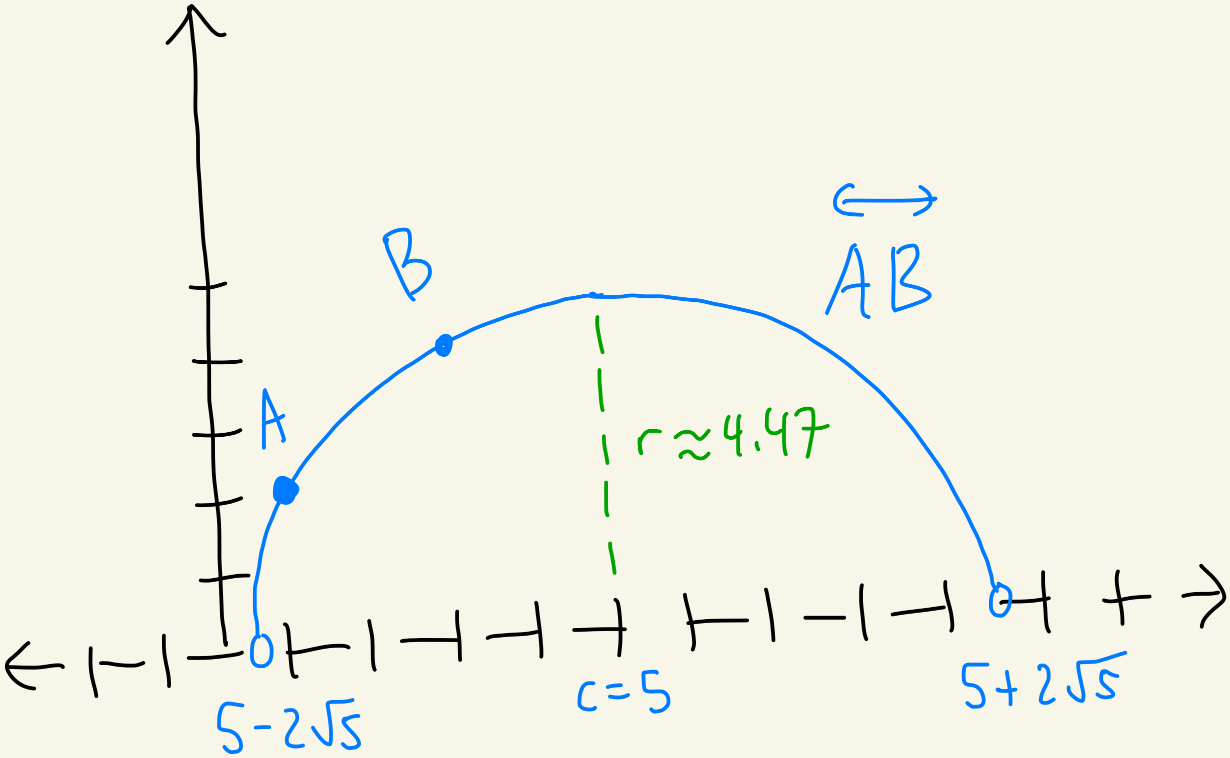
$$\Downarrow$$

$$-2c + c^2 + 5 = r^2 \quad (1)$$
$$-6c + c^2 + 25 = r^2 \quad (2)$$

(1) − (2) gives $4c - 20 = 0$. So, $c = 5$.

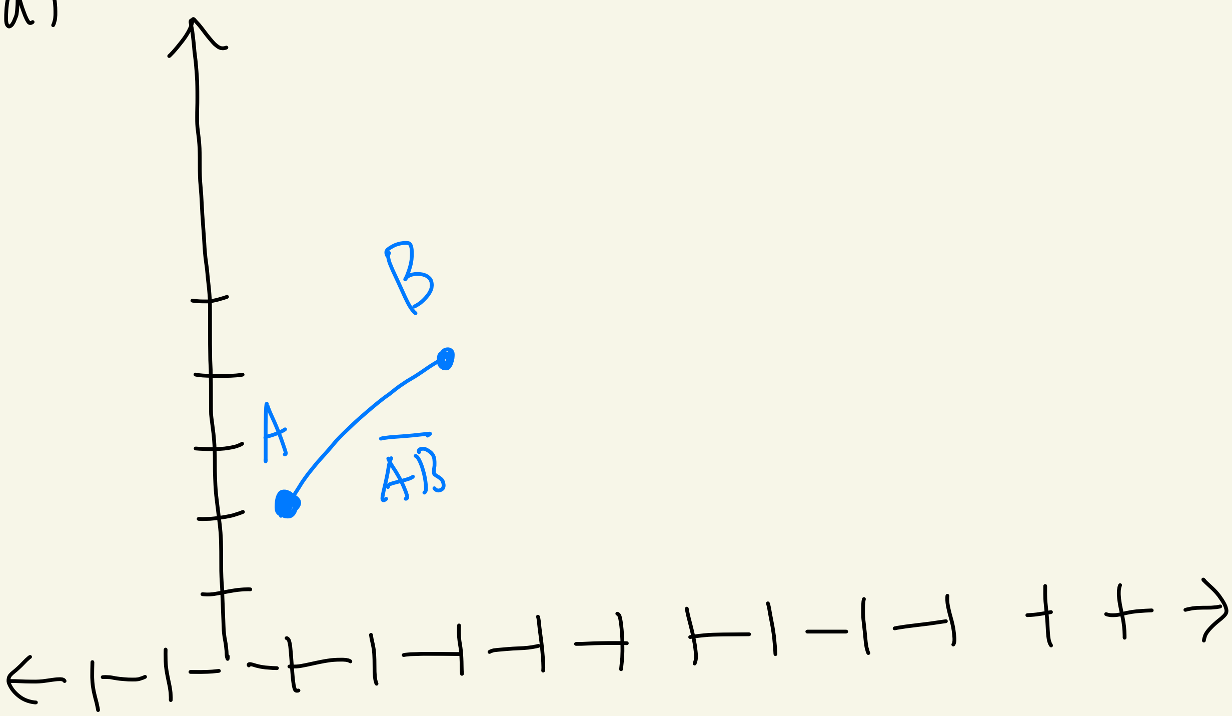
Then, from (1) we get $r = \sqrt{(1-5)^2 + 4}$
$= \sqrt{20} = 2\sqrt{5} \approx 4.47$

Thus, $A = (1,2)$, $B = (3,4)$ lie on ${}_cL_r = {}_5L_{2\sqrt{5}}$.



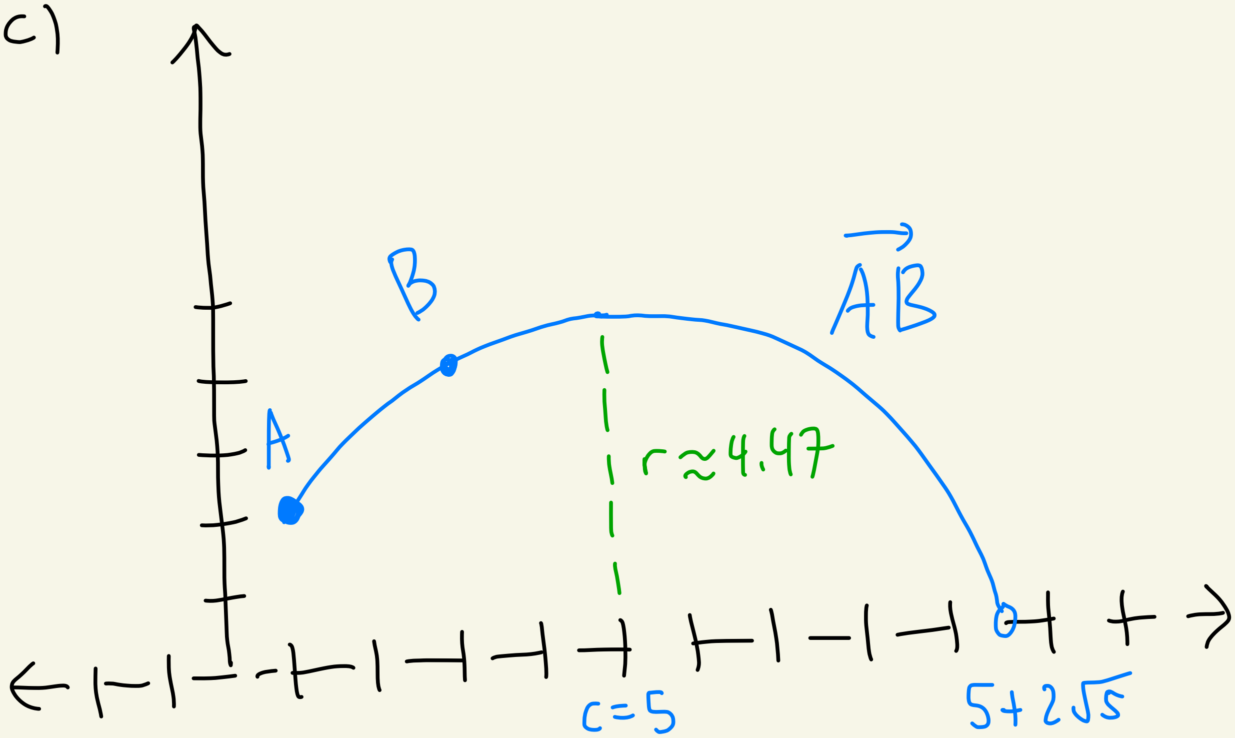
$\overleftrightarrow{AB}$
B
A
$r \approx 4.47$
$5 - 2\sqrt{5}$
$c = 5$
$5 + 2\sqrt{5}$

(a)



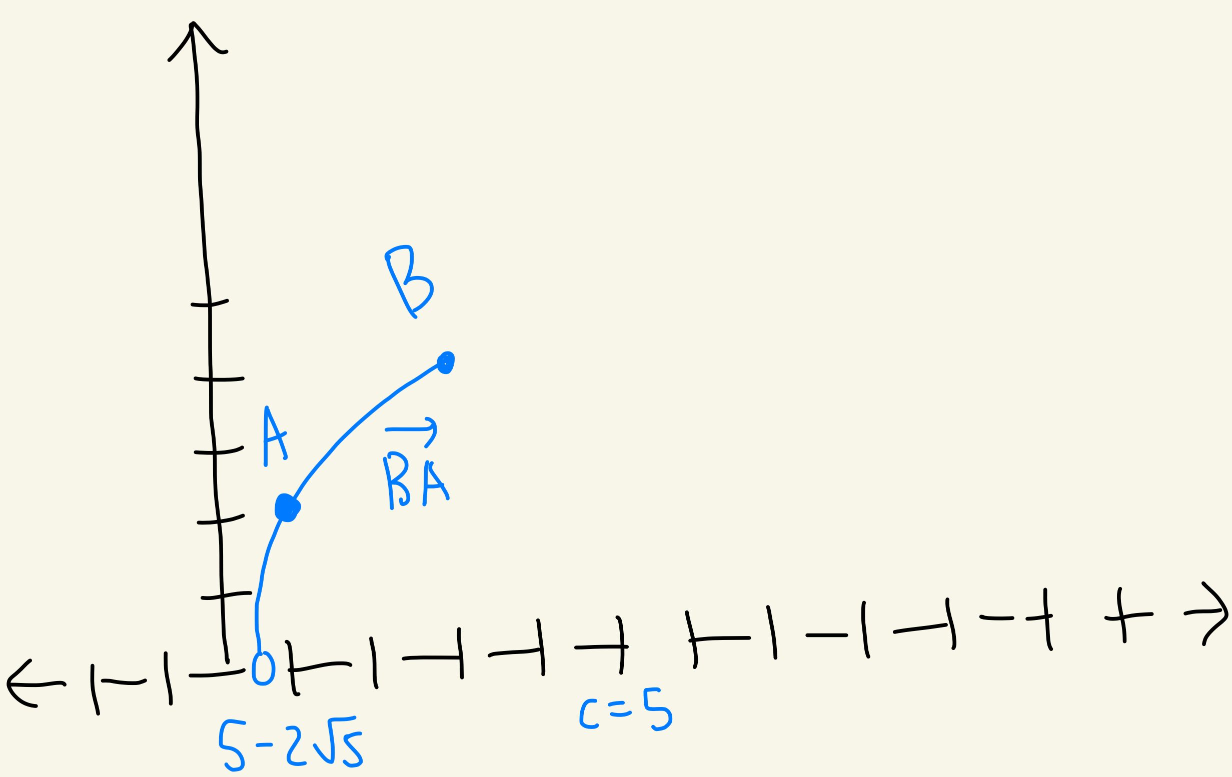
B
A
$\overline{AB}$

(c)



B
A
$\overrightarrow{AB}$
$r \approx 4.47$
$c=5$
$5+2\sqrt{5}$

(d)

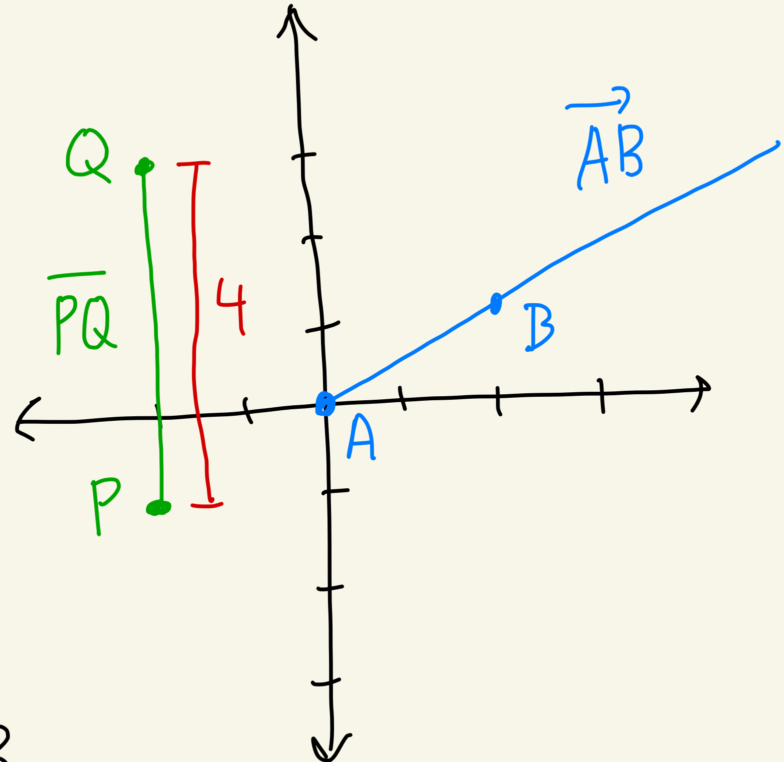


B
A
$\overrightarrow{BA}$
$5-2\sqrt{5}$
$c=5$

④ (Method 1)

$P = (-2, -1)$, $Q = (-2, 3)$

$A = (0, 0)$, $B = (2, 1)$



We have that

$$PQ = d(P, Q)$$
$$= \sqrt{(-2-(-2))^2 + (-1-3)^2}$$
$$= \sqrt{16} = 4$$

We want to find $C \in \overrightarrow{AB}$ where $AC = d(A, C) = 4$

Let $C = (x, y)$.

Since $C \in \overrightarrow{AB}$ we know from class/HW that $\boxed{C = A + t(B - A) \text{ where } t \geq 0.}$

So,

$$(x, y) = C = (0, 0) + t(2-0, 1-0) = (2t, t)$$

That is, $\boxed{x = 2t,\ y = t}$

Also since $d(A,C)=4$ we know

$$\underbrace{\sqrt{(x-0)^2+(y-0)^2}}_{d(A,C)} = 4$$

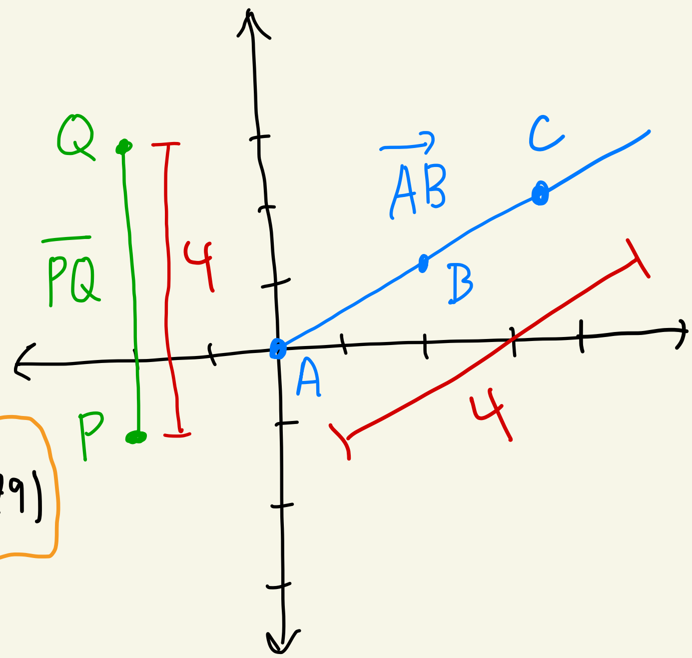
This gives $\boxed{x^2+y^2=16}$

Plug $x=2t, y=t$ into $x^2+y^2=16$
to get $4t^2+t^2=16$

So, $t^2=\frac{16}{5}$. Thus, $t=\pm\frac{4}{\sqrt{5}}$

Since $t \geq 0$ must be true from above we get $t=\frac{4}{\sqrt{5}}$

$$\boxed{\text{Thus, } C=(2t,t) = \left(\tfrac{8}{\sqrt{5}},\tfrac{4}{\sqrt{5}}\right) \approx (3.58, 1.79)}$$



Then $C \in \overrightarrow{AB}$ and
$AC = d(A,C) = 4 = d(P,Q) = PQ$

④ (Method 2)

See solution above.
You could instead do this.

$PQ = 4$ as above

Let $C = (x, y)$.

Want $4 = AC = \sqrt{(0-x)^2 + (0-y)^2} = \sqrt{x^2 + y^2}$

So need $16 = x^2 + y^2$.

The line $\overleftrightarrow{AB}$ can be described as $y = \frac{1}{2}x$.

Plug $y = \frac{1}{2}x$ into $16 = x^2 + y^2$ to get

$$16 = x^2 + \left(\tfrac{1}{2}x\right)^2$$

So, $16 = \frac{5}{4}x^2$.

So, $x^2 = \frac{64}{5}$

Thus, $x = \pm \frac{8}{\sqrt{5}}$.

Since C must be in the first quadrant we have $x = \frac{8}{\sqrt{5}}$.

Then, $y = \frac{1}{2}x = \frac{1}{2}\left(\frac{8}{\sqrt{5}}\right) = \frac{4}{\sqrt{5}}$.

So, $C = (x, y) = \left(\frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$.

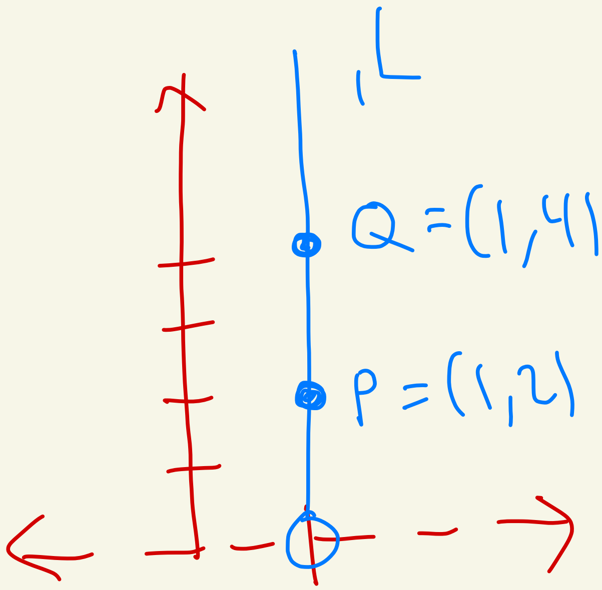
$\blacksquare$

5 In the hyperbolic plane,

let $P=(1,2)$, $Q=(1,4)$,

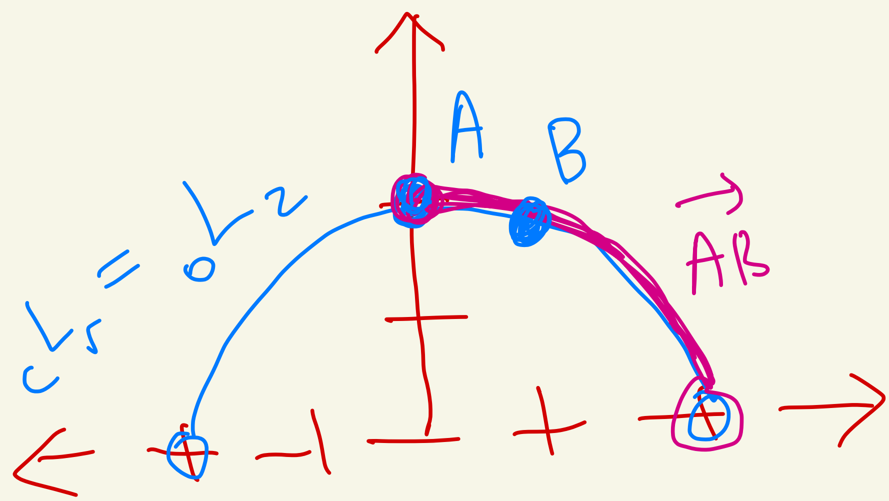
$A=(0,2)$, $B=(1,\sqrt{3})$.

Find $C \in \overrightarrow{AB}$ where $\overline{AC} \cong \overline{PQ}$.



you can calculate:

$\overleftrightarrow{AB} = {}_0L_2$



<u>Measure $\overline{PQ}$:</u>

$$d_H(P,Q) = \left|\ln\left(\frac{4}{2}\right)\right| = |\ln(2)| = \ln(2)$$

<u>Want:</u> Find $C \in \overrightarrow{AB}$ where $d_H(A,C) = \ln(2)$

Let $C=(x,y)$.

Want to solve:

$$\ln(2) = d_H(A,C)$$
$$= d_H((0,2),(x,y))$$
$$= \left|\ln\left(\frac{\frac{0-0+2}{2}}{\frac{x-0+2}{y}}\right)\right|$$
$$= \left|\ln\left(\frac{1}{\frac{x+2}{y}}\right)\right| = \left|\ln\left(\frac{y}{x+2}\right)\right|$$

So we need $\ln(2) = \pm\ln\left(\frac{y}{x+2}\right)$.

So either

$$\ln(2) = \ln\left(\frac{y}{x+2}\right) \quad \text{or} \quad \ln(2) = \underbrace{-\ln\left(\frac{y}{x+2}\right)}_{\ln\left(\frac{x+2}{y}\right)}$$

So either

$$2 = \frac{y}{x+2} \quad \text{or} \quad 2 = \frac{x+2}{y}.$$

So either

$$\boxed{y = 2x+4} \ \ 1 \quad \text{or} \quad \boxed{y = \tfrac{1}{2}x+1}, \ \ 2$$

Now plug these into $\underbrace{{}_0L_2}_{x^2+y^2=4}$ to get C

We get these two possibilities:

$$\boxed{\begin{aligned} x^2+(2x+4)^2 &= 4 \\ x^2+\left(\tfrac{1}{2}x+1\right)^2 &= 4 \end{aligned}} \quad \begin{aligned} &1 \\ &2 \end{aligned}$$

These become:

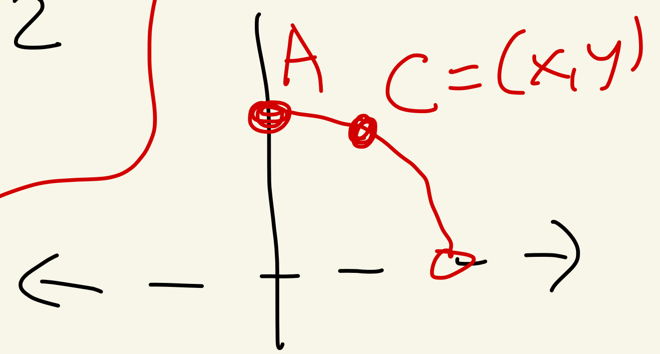
$$5x^2 + 16x + 12 = 0 \quad ①$$

$$5x^2 + 4x - 12 = 0 \quad ②$$

① becomes: $(5x+6)(x+2) = 0$

So, $x = -\frac{6}{5}$ or $x = -2$

We need C on the right side of A, so neither of these x's work.



(need $x > 0$)

② becomes: $(5x-6)(x+2) = 0$

So, $\boxed{x = \frac{6}{5}}$ or $x = -2$.

Only $x = \frac{6}{5}$ is positive.

Now plug $C = \left(\frac{6}{5}, y\right)$ into $\underbrace{{}_0L_2}_{x^2+y^2=4}$ to get:

$$\left(\frac{6}{5}\right)^2 + y^2 = 4$$

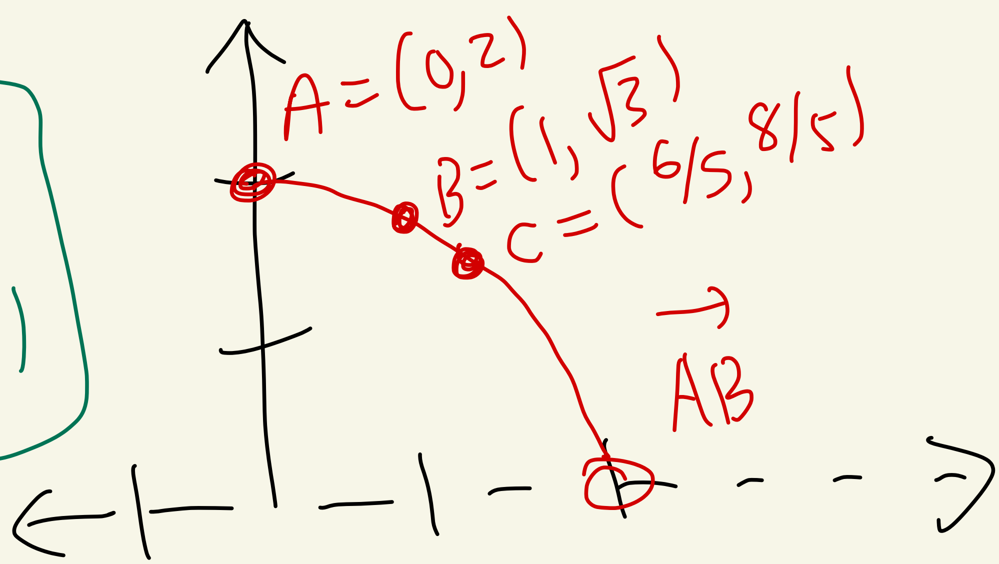
This gives $y^2 = 4 - \frac{36}{25} = \frac{100-36}{25} = \frac{64}{25}$

So, $y = \pm\sqrt{\frac{64}{25}} = \pm\frac{8}{5}$

Need $y > 0$ so we get $y = \frac{8}{5}$.

Thus, $C = \left(\frac{6}{5}, \frac{8}{5}\right)$.

We should have $d_H(A,C) = \ln(2)$



(check on next page)

Check:

$$d_H(A,C) = \left| \ln\left(\frac{\frac{0-0+2}{2}}{\frac{6/5-0+2}{8/5}} \right) \right|$$

$$= \left| \ln\left(\frac{1}{\frac{16/5}{8/5}} \right) \right| = \left| \ln\left(\frac{8}{16}\right) \right|$$

$$= \left| \ln\left(\frac{1}{2}\right) \right| = -\ln\left(\frac{1}{2}\right) = \ln(2)$$

6 Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let A and B be distinct points from $\mathscr{P}$.

(a)

$$\overline{AB} = \{ C \in \mathscr{P} \mid A - C - B \text{ or } C = A \text{ or } C = B \}$$

$$= \{ C \in \mathscr{P} \mid B - C - A \text{ or } C = B \text{ or } C = A \}$$

$$= \overline{BA}$$

$A - C - B$ iff $B - C - A$ from class

(b)

$$\overline{AB} \subseteq \overline{AB} \cup \{ C \in \mathscr{P} \mid A - B - C \} = \overrightarrow{AB}$$

So, $\overline{AB} \subseteq \overrightarrow{AB}$.

Recall from HW 4 that if $C \in \overleftrightarrow{AB}$ then either $C = A$, $C = B$, $C - A - B$, $A - C - B$, or $A - B - C$.

Thus,

$$\overrightarrow{AB} = \overline{AB} \cup \{C \in \mathscr{P} \mid A - B - C\}$$

$$= \overbrace{\{A, B\} \cup \{C \in \mathscr{P} \mid A - C - B\}}^{\overline{AB}}$$

$$\cup \{C \in \mathscr{P} \mid A - B - C\}$$

$$\subseteq \{A, B\} \cup \{C \in \mathscr{P} \mid C - A - B \text{ or } A - C - B \text{ or } A - B - C\}$$

$$= \overleftrightarrow{AB}$$

Thus, $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$.

$\boxed{(c)}$ $\overrightarrow{AB} = \overline{AB} \cup \{C \in \mathscr{P} \mid A-B-C\}$

and $\overrightarrow{BA} = \overline{BA} \cup \{C \in \mathscr{P} \mid C-B-A\}$

$= \overline{AB} \cup \{C \in \mathscr{P} \mid C-B-A\}$

$\boxed{\text{from part (a)}}$

Thus, $\overline{AB} \subseteq \overrightarrow{AB}$ and $\overline{AB} \subseteq \overrightarrow{BA}$.

So, $\boxed{\overline{AB} \subseteq \overrightarrow{AB} \cap \overrightarrow{BA}.}$

Now let's show $\overrightarrow{AB} \cap \overrightarrow{BA} \subseteq \overline{AB}$.

Let $C \in \overrightarrow{AB} \cap \overrightarrow{BA}$

Then $C \in \overrightarrow{AB}$ and $C \in \overrightarrow{BA}$.

Since $C \in \overrightarrow{AB}$ either $\boxed{C \in \overline{AB} \text{ or } A-B-C.}$ $(*)$

Since $C \in \overrightarrow{BA}$ either $\boxed{C \in \overline{BA} = \overline{AB} \text{ or } B-A-C}$ $(**)$

So $(*)$ and $(**)$ are both true.

In $(*)$ we get that $C \in \overline{AB}$ or $A-B-C$.

If $C \in \overline{AB}$ then we are done.

Suppose $A - B - C$.

From (**) we get either $C \in \overline{AB}$ or $B - A - C$.

If $C \in \overline{AB}$ then we are done.

Suppose $B - A - C$.

Then we would have $A - B - C$ and $B - A - C$.

But from HW 4 we know $A - B - C$ and $B - A - C$ cannot both happen

Thus this case cannot occur.

Thus, $C \in \overline{AB}$.

So, $\boxed{\overrightarrow{AB} \cap \overrightarrow{BA} \subseteq \overline{AB}.}$

Therefore from the above we get that $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$.

$\blacksquare$

(d)

($\subseteq$): Let $C \in \overleftrightarrow{AB}$.

Then either $C-A-B$, $C = A$, $A-C-B$, $C = B$, or $A-B-C$.

If $C-A-B$, then $B-A-C$ and so $C \in \overrightarrow{BA}$

If $C = A$, then $C \in \overline{AB} \subseteq \overrightarrow{AB}$

If $A-C-B$, then $C \in \overline{AB} \subseteq \overrightarrow{AB}$

If $C = B$, then $C \in \overline{AB} \subseteq \overrightarrow{AB}$.

If $A-B-C$, then $C \in \overrightarrow{AB}$.

Thus, from above we get $C \in \overrightarrow{AB} \cup \overrightarrow{BA}$.

($\supseteq$):

By part (b) we have $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$ and $\overrightarrow{BA} \subseteq \overleftrightarrow{AB}$.

Thus, $\overrightarrow{AB} \cup \overrightarrow{BA} \subseteq \overleftrightarrow{AB}$.

$\blacksquare$

⑦ Suppose that $A-B-C$ and $P-Q-R$ and $\overline{AB} \simeq \overline{PQ}$ and $\overline{BC} \simeq \overline{QR}$. We must show that $\overline{AC} \simeq \overline{PR}$.

<u>Goal:</u>

Since $A-B-C$ and $P-Q-R$ we know

$$\left.\begin{aligned} d(A,B)+d(B,C)&=d(A,C) \\ d(P,Q)+d(Q,R)&=d(P,R). \end{aligned}\right] \quad (*)$$

Since $\overline{AB} \simeq \overline{PQ}$ and $\overline{BC} \simeq \overline{QR}$ we know that

$$\left.\begin{aligned} d(A,B) &= d(P,Q) \\ \text{and } d(B,C) &= d(Q,R). \end{aligned}\right] \quad (**)$$

Thus,

$$\begin{aligned} d(A,C) &\overset{(*)}{=} d(A,B)+d(B,C) \\ &\overset{(**)}{=} d(P,Q)+d(Q,R) \\ &\overset{(*)}{=} d(P,R). \end{aligned}$$

So, $\overline{AC} \simeq \overline{PR}$ ∎

(8) Suppose that $A-B-C$ and $P-Q-R$ and $\overline{AB} \simeq \overline{PQ}$ and $\overline{AC} \simeq \overline{PR}$.

Goal: We must show that $\overline{BC} \simeq \overline{QR}$.

Since $A-B-C$ and $P-Q-R$ we know

$$\left.\begin{array}{l} d(A,B)+d(B,C)=d(A,C) \\ d(P,Q)+d(Q,R)=d(P,R). \end{array}\right] \quad (*)$$

Since $\overline{AB} \simeq \overline{PQ}$ and $\overline{AC} \simeq \overline{PR}$ we know that

$$\left.\begin{array}{l} d(A,B)=d(P,Q) \\ \text{and } d(A,C)=d(P,R). \end{array}\right] \quad (**)$$

Thus,

$$\begin{aligned} d(B,C) &\overset{(*)}{=} d(A,C)-d(A,B) \\ &\overset{(**)}{=} d(P,R)-d(P,Q) \\ &\overset{(*)}{=} d(Q,R). \end{aligned}$$

So, $\overline{BC} \simeq \overline{QR}$. ▨

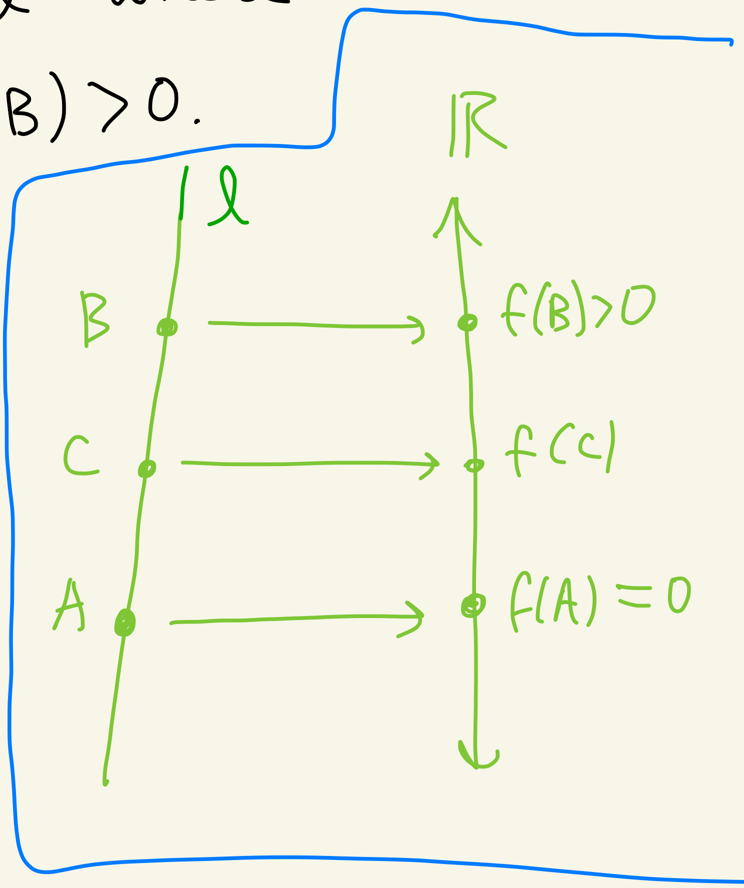
$\boxed{9(a)}$ Let $A, B, C \in \mathscr{P}$ with $A \neq B$.

Let $C \in \overrightarrow{AB}$ and $C \neq A$.

We must show that $\overrightarrow{AB} = \overrightarrow{AC}$.

Let $l = \overleftrightarrow{AB}$.

Let f be a ruler on l where
$f(A) = 0$ and $f(B) > 0$.



Then from class we have that

$$\overrightarrow{AB} = \{E \in \mathscr{P} \mid 0 \leq f(E)\}.$$

Since $C \in \overrightarrow{AB}$ we know $0 \leq f(C)$

Since $C \neq A$ and $f(A) = 0$ and f is one-to-one we know $f(C) \neq 0$.

So in fact $0 < f(C)$.

Thus, $f(A) = 0$ and $f(C) > 0$.

Since $\ell = \overleftrightarrow{AB} = \overleftrightarrow{AC}$ again from the theorem in class we get

$$\overrightarrow{AC} = \{ E \in \mathcal{P} \mid 0 \leq f(E) \}.$$

Thus, $\overrightarrow{AB} = \{ E \in \mathcal{P} \mid 0 \leq f(E) \} = \overrightarrow{AC}$.

$\blacksquare$

$\boxed{(9)(b)}$ Suppose $\overrightarrow{AB} = \overrightarrow{CD}$.

We must show that $A = C$.

Suppose that $A \neq C$.

Then, from part (a), since $C \in \overrightarrow{AB}$ and $C \neq A$

we get $\overrightarrow{AC} = \overrightarrow{AB}$.

So, $\overrightarrow{AC} = \overrightarrow{AB} = \overrightarrow{CD}$.

Thus, A, B, C, D all lie on $\ell = \overleftrightarrow{AC}$.

Let $f : \ell \to \mathbb{R}$ be a ruler centered at A where $f(A) = 0$ and $f(C) > 0$.

From class this ruler satisfies

$$\ell = \overrightarrow{AC} = \{ X \in \overleftrightarrow{AC} \mid f(X) \geq 0 \}$$

So, $f(A) < f(C)$.

We want to show that $f(C) < f(D)$.

Suppose instead that $f(A) < f(D) < f(C)$.

Let $g: \ell \to \mathbb{R}$ be given by $g(X) = -\big(f(X) - f(C)\big)$

From Topic 2 lectures we know that that g is a ruler on ℓ.

Note that $g(C) = -\big(f(C) - f(C)\big) = 0$

and $g(D) = -\big(f(D) - f(C)\big)$

$$= f(C) - f(D) > 0$$

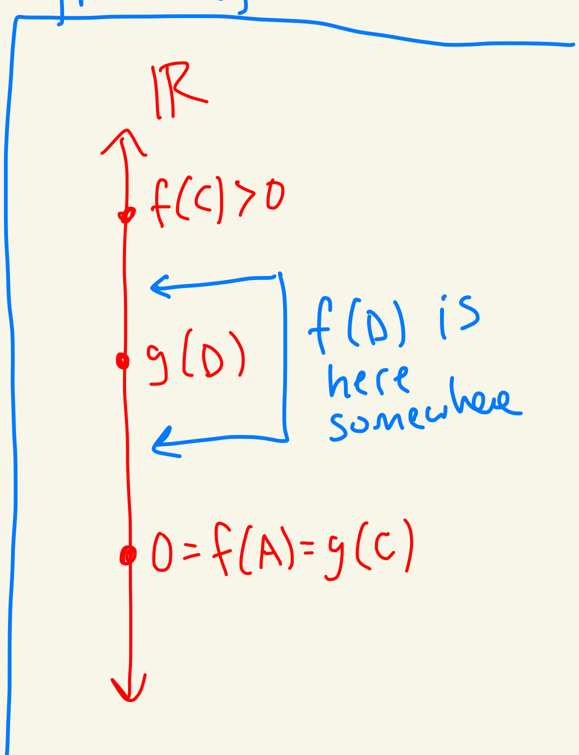
by assumption

Since g is a ruler on ℓ where $g(C) = 0$ and $g(D) > 0$, from class we know that $\overrightarrow{CD} = \{X \mid g(X) \geq 0\}$.

Since f is onto $\mathbb{R}$ there exists $E \in \mathcal{P}$ where $f(C) < f(E)$

[For example pick E where $f(E) = f(C) + 1$]

picture



Then we have

$$f(A) < f(D) < f(C) < f(E).$$

Since $f(E) > f(C) > f(A) = 0$ we know that $E \in \overrightarrow{AC}$.

However, $g(E) = -(f(E) - f(C))$

$$= f(C) - f(E) < 0$$

Thus, $E \notin \overrightarrow{CD}$. ← since $\overrightarrow{CD} = \{x \mid g(x) \geq 0\}$

Therefore, $E \in \overrightarrow{AC}$ and $E \notin \overrightarrow{CD}$.

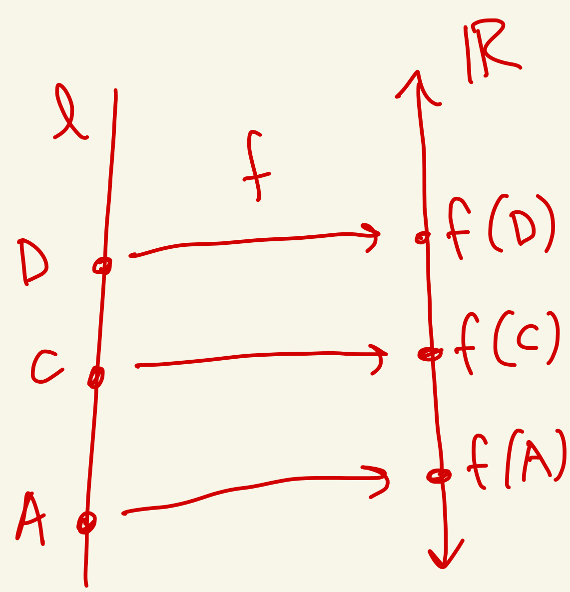
But $\overrightarrow{AC} = \overrightarrow{CD}$.

So no such E exists.

Contradiction.

Therefore, $f(C) < f(D)$,

So, $f(A) < f(C) < f(D)$.



Let $h: \ell \to \mathbb{R}$ be given by

$$h(x) = f(x) - f(c)$$

By topic 2, h is a ruler on ℓ.

Then, $h(C) = f(c) - f(c) = 0$

and $h(D) = f(D) - f(c) > 0$

since $f(D) > f(C)$

So,

$$\overrightarrow{CD} = \{x \in \wp \mid h(x) \geq 0\}$$

since $f(A) < f(C)$

But, $h(A) = f(A) - f(c) < 0$

So, $A \notin \overrightarrow{CD}$.

However $A \in \overrightarrow{AB}$ and $\overrightarrow{AB} = \overrightarrow{CD}$.

Contradiction.

Thus, the original assumption that $A \neq C$ can't happen

And $A = C$. $\boxed{/\!/\!/}$

10 (a)

Let $S = \{ C \in \mathbb{R}^2 \mid C = A + t(B-A) \text{ where } 0 \leq t \leq 1 \}$.

We must show that $\overline{AB} = S$.

$\boxed{\overline{AB} \subseteq S}$:

Let $C \in \overline{AB}$.

Then either $C = A$ or $C = B$ or $A - C - B$.

If $C = A$, then set $t = 0$ and we get $C = A = A + 0(B-A)$.

So, if $C = A$, then $C \in S$.

If $C = B$, then set $t = 1$ and we get $C = B = A + 1 \cdot (B-A)$

So if $C = B$, then $C \in S$.

Suppose $A - C - B$.

By HW 4 there exists t with $0 < t < 1$ where $C = A + t(B-A)$.

Then in this case also we have that $C \in S$.

So, $\overline{AB} \subseteq S$.

$\boxed{S \subseteq \overline{AB}}$: Suppose $C \in S$.

Then, $C = A + t(B-A)$ where $0 \leq t \leq 1$.

If $t = 0$, then $C = A + 0(B-A) = A \in \overline{AB}$

If $t = 1$, then $C = A + 1 \cdot (B-A) = B \in \overline{AB}$.

Suppose that $0 < t < 1$.

Then by HW 4, we have $A - C - B$.

So in this last case $C \in \overline{AB}$.

Thus, $S \subseteq \overline{AB}$.

Since $\overline{AB} \subseteq S$ and $S \subseteq \overline{AB}$ we have that $\overline{AB} = S$.

∎

$\boxed{10 (b)}$ Let

$S = \{ C \in \mathcal{P} \mid C = A + t(B-A) \text{ where } 0 \leq t \}$.

We must show that $\overrightarrow{AB} = S$.

We must show $\overrightarrow{AB} \subseteq S$ and $S \subseteq \overrightarrow{AB}$.

We prove

$\boxed{\overrightarrow{AB} \subseteq S}$: Let $C \in \overrightarrow{AB}$.

Then, $C \in \overleftrightarrow{AB}$.

So, from class we know that

$$C = A + t(B-A) \text{ where } t \in \mathbb{R}. \quad (*)$$

We must show that $t \geq 0$.

Suppose instead that $t < 0$.

We show this leads to a contradiction.

Case 1: Suppose $\overleftrightarrow{AB} = L_d$ for some $d \in \mathbb{R}$.

Let $A = (d, y_a)$, $B = (d, y_b)$, $C = (d, y_c)$

So, $(*)$ gives

$$(d, y_c) = (d, y_a + t(y_b - y_a))$$

Let $f : L_d \to \mathbb{R}$ be the standard ruler, that is $f(d,y) = y$.

Then applying f to the above gives

$$y_c = y_a + t\,(y_b - y_a).$$

f gives us two options:

Either $f(A) < f(B)$
or $f(B) < f(A)$.

That is either

$$y_a < y_b$$

or $$y_b < y_a.$$

Case 1 (i): Suppose $y_a < y_b$.

Then, $y_c - y_a = \underbrace{t}_{<0}\,(\underbrace{y_b - y_a}_{>0}) < 0$

So, $y_c < y_a$.

Then $y_c < y_a < y_b$.

So, $f(C) < f(A) < f(B)$.

Then, $C - A - B$.

But $C \in \overrightarrow{AB}$, so either $C = A$, $C = B$, or $A - C - B$ or $A - B - C$.

This conflicts with $C - A - B$ by HW 4.

Thus, we get a contradiction in case 1(i).

<u>case 1(ii)</u>: Suppose $y_b < y_a$.

Then, $y_c - y_a = \underbrace{t}_{<0}\underbrace{(y_b - y_a)}_{<0} > 0$.

So, $y_c > y_a$.

Then, $y_b < y_a < y_c$.

So, $f(B) < f(A) < f(C)$.

Thus, $B - A - C$.

Thus, $C - A - B$.

But $C \in \overrightarrow{AB}$, so either $C = A$, $C = B$, or $A - C - B$ or $A - B - C$.

This conflicts with $C - A - B$ by HW 4.

Thus, we get a contradiction in case 1(ii).

Therefore, in summary we get a contradiction in both parts of case 1.

Case 2: Suppose $\ell = L_{m,d}$.

If you do the same arguments as case 1, but use the standard ruler for $L_{m,d}$ then you will get a contradiction also.

Thus, case 1 and case 2 both give contradictions.

Therefore, $t \geq 0$ must be true.

Thus, $C \in S$.

So, $\overrightarrow{AB} \subseteq S$.

$\boxed{S \subseteq \overrightarrow{AB}}$: Let $C \in S$.

Then, $C = A + t(B-A)$ where $t \geq 0$.

We need to show that $C \in \overrightarrow{AB}$.

Let $\ell = \overleftrightarrow{AB}$

I'll show this for the case when $\ell = L_d$ for some $d \in \mathbb{R}$.

You can try a similar proof for $\ell = L_{m,d}$.

you do if want extra practice

Suppose that $\ell = L_d$ for some $d \in \mathbb{R}$.

Let $A = (d, y_a)$, $B = (d, y_b)$, $C = (d, y_c)$.

Then $C = A + t(B - A)$ becomes

$$\boxed{(d, y_c) = (d, y_a + t(y_b - y_a))} \qquad (*)$$

Let $f : \ell \to \mathbb{R}$ be the standard ruler

where $f(d, y) = y$.

<u>Case (i):</u> Suppose $y_a < y_b$, that is $f(A) < f(B)$

Let $g : \ell \to \mathbb{R}$ be

given by $g(x) = f(x) - f(A)$.

By topic 3, g is a ruler on ℓ.

Also, $g(A) = f(A) - f(A) = 0$

and $g(B) = f(B) - f(A) > 0$.

Since $g(A) = 0$ and $g(B) > 0$ we know

$\overrightarrow{AB} = \{ x \in \ell \mid g(x) \geq 0 \}$.

topic 5 notes

Note that $\overbrace{g(d,y)}^{g(x)} = \overbrace{f(d,y)}^{f(x)} - \overbrace{f(d,y_a)}^{f(A)}$

$$= y - y_a$$

Thus applying g to $(*)$ gives

$$y_c - y_a = \left[y_a + t(y_b - y_a)\right] - y_a.$$

So, $\overbrace{y_c - y_a}^{g(C)} = t\overbrace{(y_b - y_a)}^{g(B)}.$

So, $g(c) = \underbrace{t}_{\geq 0} \cdot \underbrace{g(B)}_{>0} \geq 0.$

Thus, $C \in \overrightarrow{AB}$.

$\boxed{\text{since } \overrightarrow{AB} = \{x \in \ell \mid g(x) \geq 0\}}$

So case (i) is done.

<u>Case (ii)</u> Suppose $y_b < y_a$, that is $f(B) < f(A)$

Let $g : \ell \to \mathbb{R}$ be given by $g(x) = -(f(x) - f(A))$

$$= f(A) - f(x)$$

By topic 3, g is a ruler on ℓ.

Also, $g(A) = f(A) - f(A) = 0$

and $g(B) = f(A) - f(B) > 0$

Since $g(A) = 0$ and $g(B) > 0$ we know

$$\overrightarrow{AB} = \{x \in \ell \mid g(x) \geq 0\}.$$

topic 5 notes

Note that $\overbrace{g(d,y)}^{g(x)} = \overbrace{f(d,y_a)}^{f(A)} - \overbrace{f(d,y)}^{f(x)}$

$$= y_a - y$$

Thus applying g to $(*)$ gives

$$y_a - y_c = y_a - \left[y_a + t(y_b - y_a)\right]$$

Therefore, $\overbrace{y_a - y_c}^{g(C)} = t\overbrace{(y_a - y_b)}^{g(B)}$

So, $g(C) = \underbrace{t}_{\geq 0} \cdot \underbrace{g(B)}_{>0} \geq 0.$

Thus, $C \in \overrightarrow{AB}$.

Therefore, case (ii) is done.

In both cases we get $C \in \overrightarrow{AB}$.

Thus, $S \subseteq \overrightarrow{AB}$.

Therefore, $S = \overrightarrow{AB}$.

$\blacksquare$

(11) Let $A=(x_1,y_1)$ and $B=(x_2,y_2)$ with $x_1 < x_2$.

Suppose A and B both lie on ${}_cL_r$.

Suppose that $C=(x,y)$ lies on ${}_cL_r$ and that $x_1 < x < x_2$.

We must show that $C \in \overline{AB}$.

Since $C \neq A$ and $C \neq B$ this comes down to showing that $A-C-B$

We know A,B,C are distinct points all lying on ${}_cL_r$.

Let $f: {}_cL_r \longrightarrow \mathbb{R}$ be the standard ruler given by $f(a,b) = \ln\left(\frac{a-c+r}{b}\right)$.

Recall that $f^{-1}: \mathbb{R} \to {}_cL_r$ is given by

$$f^{-1}(t) = (c + r\tanh(t), r\,\mathrm{sech}(t))$$

Let $f(A) = t_1$, $f(B) = t_2$, $f(C) = t$.

Then

$$\left.\begin{aligned} x_1 &= c + r\tanh(t_1) \\ x_2 &= c + r\tanh(t_2) \\ x &= c + r\tanh(t) \end{aligned}\right\} \quad \begin{aligned} &\text{Here} \\ &A = (x_1, y_1) \\ &B = (x_2, y_2) \\ &C = (x, y) \end{aligned}$$

We are given that $x_1 < x < x_2$.

Since $\tanh(s)$ is an increasing function this implies that $c + r\tanh(s)$ is an increasing function.

So, $x_1 < x < x_2$ implies $t_1 < t < t_2$.

Thus, $f(A) < f(C) < f(B)$

Therefore $A - C - B$.

$\blacksquare$